International J.Math. Combin. Vol.3 (2010), 98-107

Tulgeity of Line, Middle and Total Graph of Wheel Graph Families

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Abstract: Tulgeity $\tau(G)$ is the maximum number of disjoint, point induced, non acyclic subgraphs contained in G. In this paper we find the tulgeity of line, middle and total graph of wheel graph, Gear graph and Helm graph.

Key Words: Tulgeity, Smarandache partition, line graph, middle graph, total graph and wheel graph.

AMS(2000): 05C70, 05C75, 05C76

§1. Introduction

The *point partition number* [4] of a graph G is the minimum number of subsets into which the point-set of G can be partitioned so that the subgraph induced by each subset has a property P. Dual to this concept of point partition number of graph is the maximum number of subsets into which the point-set of G can be partitioned such that the subgraph induced by each subset does not have the property P. Define the property P such that a graph G has the property P if G contains no subgraph which is homeomorphic to the complete graph K_3. Now the point partition number and dual point partition number for the property P is referred to as point arboricity and tulgeity of G respectively. Equivalently the tulgeity is the maximum number of vertex disjoint subgraphs contained in G so that each subgraph is not acyclic. This number is called the tulgeity of G denoted by $\tau(G)$. Also, $\tau(G)$ can be defined as the maximum number

[1]Received June 28, 2010. Accepted September 18, 2010.

of disjoint cycles in G. The formula for tulgeity of a complete bipartite graph is given in [5]. The problems of Nordhaus-Gaddum type for the dual point partition number are investigated in [3].

Let P be a graph property and G be a graph. If there exists a partition of G with a partition set pair $\{H, T\}$ such that the subgraph induced by a subset in H has property P, but the subgraph induced in T has no property P, then we say G possesses the *Smarandache partition*. Particularly, let $H = \emptyset$ or $T = \emptyset$, we get the conception of point partition or its dual.

All graphs considered in this paper are finite and contains no loops and no multiple edges. Denote by $[x]$ the greatest integer less than or equal to x, by $|S|$ the cardinality of the set S, by $E(G)$ the edge set of G and by K_n the complete graph on n vertices. p_G and q_G denotes the number of vertices and edges of the graph G. The other notations and terminology used in this paper can be found in [6].

Line graph $L(G)$ of a graph G is defined with the vertex set $E(G)$, in which two vertices are adjacent if and only if the corresponding edges are adjacent in G. Since $\tau(G) \leq \left[\frac{p}{3}\right]$, it is obvious that $\tau(L(G)) \leq \left[\frac{q}{3}\right]$. However for complete graph K_p, $\tau(K_p) = \left[\frac{p}{3}\right]$.

Middle graph $M(G)$ of a graph G is defined with the vertex set $V(G) \cup E(G)$, in which two elements are adjacent if and only if either both are adjacent edges in G or one of the elements is a vertex and the other one is an edge incident to the vertex in G. Clearly $\tau(M(G)) \leq \left[\frac{p+q}{3}\right]$.

Total graph $T(G)$ of a graph G defined with the vertex set $V(G) \cup E(G)$, in which two elements are adjacent if and only if one of the following holds true (i) both are adjacent edges or vertices in G (ii) one is a vertex and other is an edge incident to it in G.

§2. Basic Results

We begin by presenting the results concerning the tulgeity of a graph.

Theorem 2.1([5]) *For any graph G, $\tau(G) = \sum \tau(C) \leq \tau(B)$, where the sums being taken over all components C and blocks B of G, respectively.*

Theorem 2.2([5]) *For the complete n-partite graph $G = K(p_1, p_2, ..., p_n), 1 \leq p_1 \leq p_2 \leq ..... \leq p_n$ and $\sum p_i = p$, $\tau(G) = min\left(\left[\frac{1}{2}\sum_0^{n-1} p_i\right], [p/3]\right)$, where $p_0 = 0$.*

We have derived [1] the formula to find the tulgeity of the line graph of complete and complete bigraph.

Theorem 2.3([1]) $\tau(L(K_n)) = \left[\frac{n(n-1)}{6}\right]$.

Theorem 2.4([1]) $\tau(L(K_{m,n}) = \left[\frac{mn}{3}\right]$.

Also, we have derived an upper bound for the tulgeity of line graph of any graph and characterized the graphs for which the upper bound equal to the tulgeity.

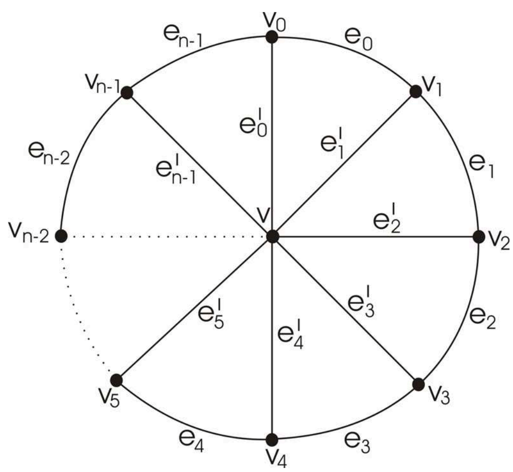
Theorem 2.5([1]) *For any graph G, $\tau(L(G)) \leq \sum\limits_i \left[\frac{\deg v_i}{3}\right]$ where $\deg v_i$ denotes the degree of the vertex v_i and the the summation taken over all the vertices of G.*

Theorem 2.6([1]) *If G is a tree and for each pair of vertices (v_i, v_j) with $\deg v_i$, $\deg v_j > 2$, if there exist a vertex v of degree 2 on $P(v_i, v_j)$ then $\tau(L(G)) \leq \sum\limits_i \left[\frac{\deg v_i}{3}\right]$.*

We have derived the results to find the tulgeity of Knödel graph, Prism graph and their line graph in [2].

§3. Wheel Graph

The wheel graph W_n on $n+1$ vertices is defined as $W_n = C_n + K_1$ where C_n is a n-cycle. Let $V(W_n) = \{v_i : 0 \leq i \leq n-1\} \cup \{v\}$ and $E(W_n) = \{e_i = v_i v_{i+1} : 0 \leq i \leq n-1, \text{subscripts modulo } n\} \cup \{e'_i = vv_i : 0 \leq i \leq n-1\}$.



Wheel graph W_n

Figure 3.1

Theorem 3.1 *The Tulgeity of the line graph of W_n,*

$$\tau(L(W_n)) = \left[\frac{2n}{3}\right].$$

Proof By the definition of line graph, $V(L(W_n)) = E(W_n) = \{e_i : 0 \leq i \leq n-1, \text{subscripts modulo n}\} \cup \{e'_i : 0 \leq i \leq n-1\}$. Let

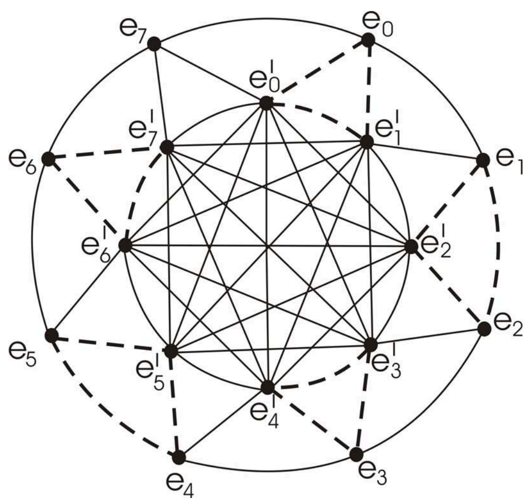
$$\mathbb{C} = \left\{e_i e'_i e'_{i+1} : i = 3(k-1), 1 \leq k \leq \left[\frac{n}{3}\right]\right\}$$

and

$$\mathbb{C}' = \left\{e_i e_{i+1} e'_{i+1} : i = 3k-2, 1 \leq k \leq \left[\frac{n}{3}\right]\right\}$$

be a collection of 3-cycles of $L(W_n)$. Clearly the cycles of $\mathbb{C}$ and $\mathbb{C}'$ are vertex disjoint and if $V(\mathbb{C})$ and $V(\mathbb{C}')$ denotes the set of vertices belonging to the cycles of $\mathbb{C}$ and $\mathbb{C}'$ respectively then $V(\mathbb{C}) \cap V(\mathbb{C}') = \emptyset$. Hence $\tau(L(W_n)) \geq |\mathbb{C}| + |\mathbb{C}'| = 2\left[\frac{n}{3}\right]$.

If $n \equiv 0$ or $1(mod\ 3)$, then $2\left[\frac{n}{3}\right] = \left[\frac{2n}{3}\right]$. Hence $\tau(L(W_n)) \geq \left[\frac{2n}{3}\right]$. If $n \equiv 2(mod\ 3)$, then $\left[\frac{2n}{3}\right] = 2\left[\frac{n}{3}\right] + 1$. In this case $e'_{n-2}, e'_{n-1}, e_{n-2}, e_{n-1} \notin V(\mathbb{C}) \cup V(\mathbb{C}')$ and the set $\{e'_{n-2}, e'_{n-1}, e_{n-2}\}$ induces a 3-cycle. Hence if $n \equiv 2(mod\ 3)$, $\tau(L(W_n)) \geq 2\left[\frac{n}{3}\right] + 1 = \left[\frac{2n}{3}\right]$. Therefore in both the cases $\tau(L(W_n)) \geq \left[\frac{2n}{3}\right]$. Also since $|V(L(W_n))| = 2n$, $\tau(L(W_n)) \leq \left[\frac{2n}{3}\right]$. Hence $\tau(L(W_n)) = \left[\frac{2n}{3}\right]$. □



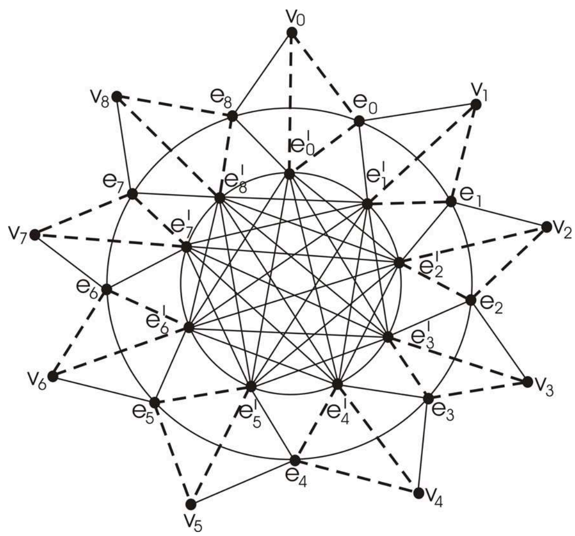
$L(W_8)$ and its vertex disjoint cycles

Figure 3.2

Theorem 3.2 *The Tulgeity of the middle graph of* W_n, $\tau(M(W_n)) = n$.

Proof By the definition of middle graph, $V(M(W_n)) = V(W_n) \cup E(W_n)$, in which for any two elements $x, y \in V(M(W_n))$, $xy \in E(M(W_n))$ if and only if any one of the following holds. (i) $x, y \in E(W_n)$ such that x and y are adjacent in W_n, (ii) $x \in V(W_n)$, $y \in E(W_n)$ or $x \in E(W_n)$, $y \in V(W_n)$ such that x and y are incident in W_n. Since $V(M(W_n)) = V(W_n) \cup E(W_n)$, $|V(M(W_n))| = n + 1 + 2n = 3n + 1$ and hence $\tau(M(W_n)) \leq \left[\frac{3n+1}{3}\right] = n$. Let $\mathbb{C} = \{C_i = v_i e_i e'_i : 0 \leq i \leq n-1\}$ be the collection of cycles of $M(W_n)$. Clearly the cycles of $\mathbb{C}$ are vertex disjoint and $|\mathbb{C}| = n$. Hence $\tau(M(W_n)) \geq n$ which implies $\tau(M(W_n)) = n$. □

By the definition of total graph $V(M(W_n)) = V(T(W_n))$ and $E(M(W_n)) \subset E(T(W_n))$. Also since $\tau(M(W_n)) = n = \left[\frac{1}{3} p_{M(W_n)}\right]$, we conclude the following result.



$M(W_9)$ and its vertex disjoint cycles

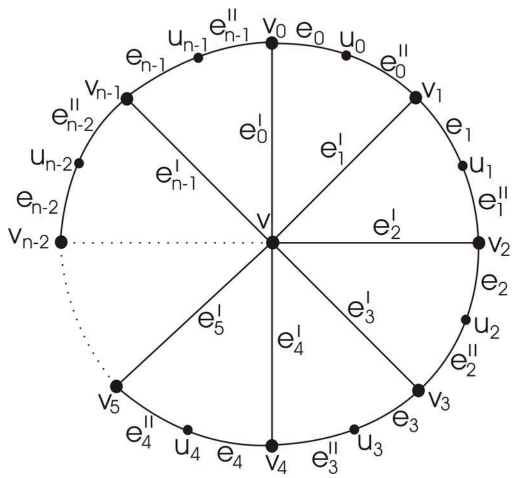
Figure 3.3

Theorem 3.3 *For any wheel graph W_n, the tulgeity of its total graph,*

$$\tau(T(W_n)) = \tau(M(W_n)) = n.$$

§4. Gear Graph

The gear graph is a wheel graph with vertices added between pair of vertices of the outer cycle. The gear graph G_n has $2n+1$ vertices and $3n$ edges.



Gear Graph G_n

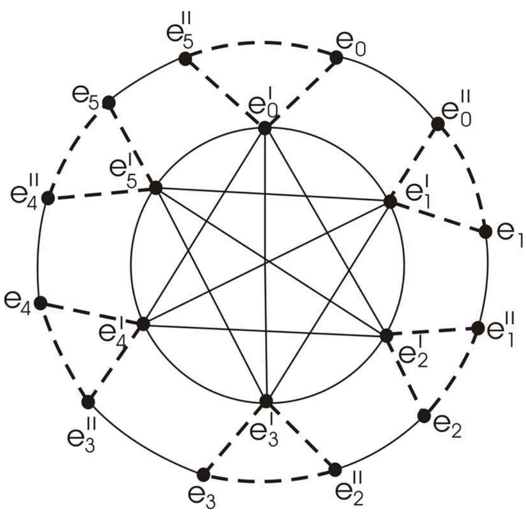
Figure 4.1

Let $V(G_n) = \{v_i : 0 \le i \le n-1\} \cup \{u_i : 0 \le i \le n-1\} \cup \{v\}$ and $E(G_n) = \{e_i = v_i u_i : 0 \le i \le n-1\} \cup \{e'_i = vv_i : 0 \le i \le n-1\} \cup \{e''_i = u_i v_{i+1} : 0 \le i \le n-1, \text{subscripts modulo } n\}$.

Theorem 4.1 *For any gear graph G_n, the tulgeity of its line graph,*

$$\tau(L(G_n)) = n.$$

Proof By the definition of line graph, $V(L(G_n)) = E(G_n)$, in which the set of vertices of $L(G_n)$, $\{e'_i : 0 \le i \le n-1\}$ induces a clique of order n. Also for each i, $(0 \le i \le n-1)$, the set $\{e''_i e'_{i+1} e_{i+1} : \text{subscripts modulo } n\}$ induces vertex disjoint clique of order 3. Let $\mathbb{C} = \{e''_i e'_{i+1} e_{i+1} : 0 \le i \le n-1, \text{subscripts modulo } n\}$ be the set of cycles of $L(G_n)$. It is clear that the cycles of $\mathbb{C}$ are vertex disjoint and $|\mathbb{C}| = n$ therefore $\tau(L(G_n)) \ge n$. Also, since $p_{L(G_n)} = q_{G_n} = 3n$, $\tau(L(G_n)) \le \left[\frac{3n}{3}\right] = n$. Hence $\tau(L(G_n)) = n$. □



$L(G_6)$ and its vertex disjoint cycles

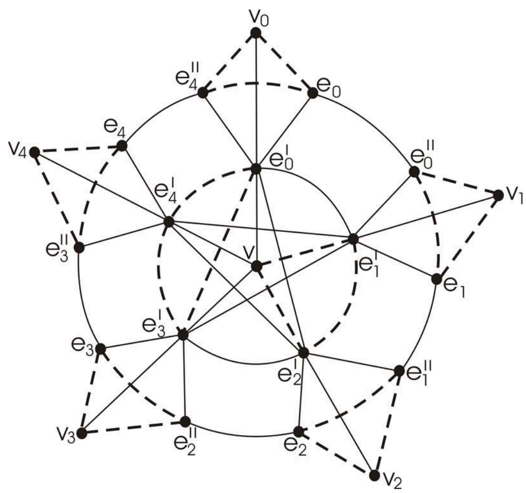
Figure 4.2

Theorem 4.2 *For any gear graph G_n, the tulgeity of its middle graph,*

$$\tau(M(G_n)) = \left[\frac{4n+1}{3}\right].$$

Proof Since $p_{M(G_n)} = p_{G_n} + q_{G_n} = (n+1) + 3n = 4n+1$, $\tau(M(G_n)) = \left[\frac{4n+1}{3}\right]$. By the definition of middle graph $V(M(G_n)) = V(G_n) \cup E(G_n)$, in which the set of vertices $\{e'_i : 0 \le i \le n-1\} \cup \{v\}$ induces a clique K_{n+1} of order $n+1$ and for each i, $(0 \le i \le n-1)$ the set $\{e''_i e'_{i+1} e_{i+1} v_{i+1} : \text{subscripts modulo } n\}$ induces a clique of order 4. From these cliques we form the set of cycles of $M(G_n)$. Let $\mathbb{C}$={set of vertex disjoint 3-cycles of the clique K_{n+1}} and $\mathbb{C}' = \{e''_i e'_{i+1} e_{i+1} v_{i+1} : 0 \le i \le n-1, \text{subscripts modulo } n\}$. Clearly $V(\mathbb{C}) \cap V(\mathbb{C}') = \emptyset$

and hence the cycles of $\mathbb{C}$ and $\mathbb{C}'$ are vertex disjoint. Also $|\mathbb{C}| = \left[\dfrac{n+1}{3}\right]$ and $|\mathbb{C}'| = n$. Hence $\tau(M(G_n)) \geq |\mathbb{C}| + |\mathbb{C}'| = \left[\dfrac{4n+1}{3}\right]$. Therefore $\tau(M(G_n)) = \left[\dfrac{4n+1}{3}\right]$. □



$M(G_5)$ and its vertex disjoint cycles

Figure 4.3

By the definition of total graph $V(M(G_n)) = V(T(G_n))$ and $E(M(G_n)) \subset E(T(G_n))$. Also since $\tau(M(G_n)) = \left[\dfrac{4n+1}{3}\right] = \left[\dfrac{1}{3}p_{M(G_n)}\right]$, we conclude the following result.

Theorem 4.3 *For any gear graph G_n, the tulgeity of its middle graph,*

$$\tau(M(G_n)) = \tau(T(G_n)) = \left[\frac{4n+1}{3}\right].$$

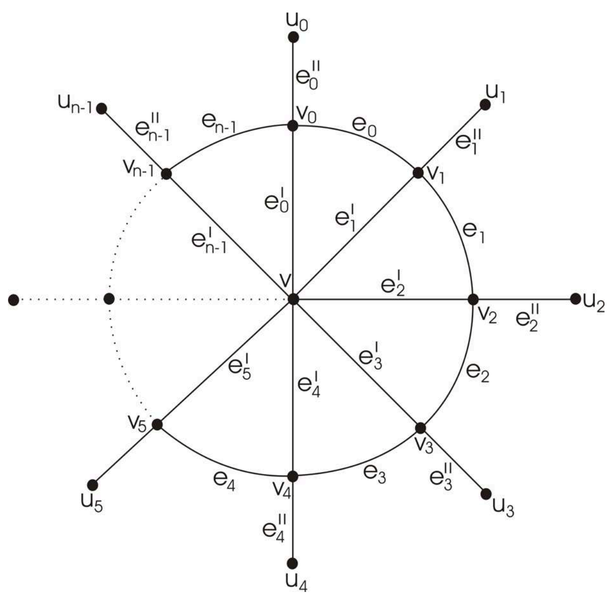
§5. Helm Graph

The helm graph H_n is the graph obtained from an n-wheel graph by adjoining a pendant edge at each node of the cycle.

Let $V(H_n) = \{v\} \cup \{v_i : 0 \leq i \leq n-1\} \cup \{u_i : 0 \leq i \leq n-1\}$, $E(H_n) = \{e_i = v_i v_{i+1} : 0 \leq i \leq n-1, \text{subscript modulo } n\} \cup \{e'_i = vv_i : 0 \leq i \leq n-1\} \cup \{e''_i = v_i u_i : 0 \leq i \leq n-1\}$.

Theorem 5.1 *For any helm graph H_n, $\tau(L(H_n)) = n$.*

Proof By the definition of line graph, $V(L(H_n)) = \{e_i : 0 \leq i \leq n-1\} \cup \{e'_i : 0 \leq i \leq n-1\} \cup \{e''_i : 0 \leq i \leq n-1\}$. Since e_i, e'_i and e''_i $(0 \leq i \leq n-1)$ are adjacent edges in H_n, $\{e_i, e'_i, e''_i\}$ induces a 3-cycle in $L(H_n)$ for each i, $(0 \leq i \leq n-1)$. Let $\mathbb{C} = \{e_i e'_i e''_i : 0 \leq i \leq n-1\}$ be the set of these cycles. Clearly $\mathbb{C}$ contains vertex disjoint cycles of $L(H_n)$ and $|\mathbb{C}| = n$. Hence $\tau(L(H_n)) \geq n$. Also since $|V(L(H_n))| = 3n, \tau(L(H_n)) \leq n$. Therefore $\tau(L(H_n)) = n$. □



Helm Graph H_n

Figure 5.1

Theorem 5.2 *The Tulgeity of the middle graph of the helm graph H_n, is given by*

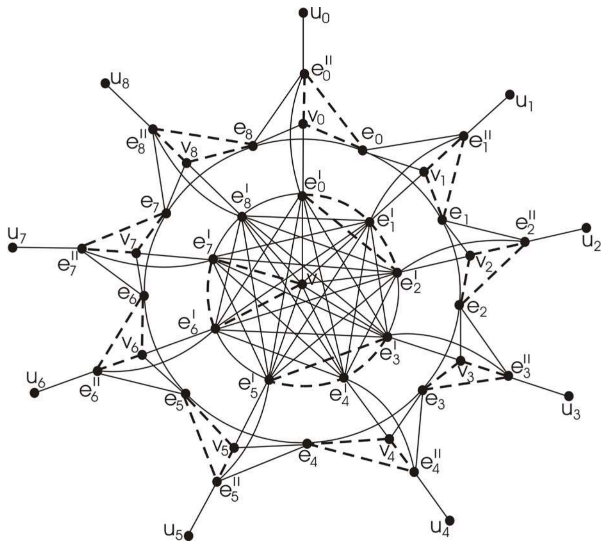
$$\tau(M(H_n)) = \left[\frac{4n+1}{3}\right].$$

Proof By the definition of middle graph, $V(M(H_n)) = V(H_n) \cup E(H_n)$, in which for each i, $(0 \leq i \leq n-1)$, the set of vertices $\{e_i, e_{i+1}, e'_{i+1}, e''_{i+1}, v_{i+1}$: subscript modulo $n\}$ induce a clique of order 5. Also $\{e'_i : 0 \leq i \leq n-1\} \cup \{v\}$ induces a clique of order $n+1$ (say K_{n+1}). Since $deg\, u_i = 1$ for each i, $(0 \leq i \leq n-1)$ in $M(H_n)$ $\tau(M(H_n)) = \tau(M(H_n) - \{u_i : 0 \leq i \leq n-1\})$. Hence $\tau(M(H_n)) \leq \left[\frac{1}{3}(|E(H_n)| + |V(H_n)| - n)\right] = \left[\frac{4n+1}{3}\right]$. Consider the collection $\mathbb{C}$ of cycles of $M(H_n)$, $\mathbb{C} = \{v_i e_i e''_i : 0 \leq i \leq n-1\}$. Each cycle of $\mathbb{C}$ are vertex disjoint and $|\mathbb{C}| = n$. Also the cycles of $\mathbb{C}$ are vertex disjoint from the cycles of the clique K_{n+1}. Hence $\tau(M(H_n)) \geq |\mathbb{C}| + \left[\frac{n+1}{3}\right] = \left[\frac{4n+1}{3}\right]$. Therefore $\tau(M(H_n)) = \left[\frac{4n+1}{3}\right]$. □

Theorem 5.3 *Tulgeity of total graph of helm graph H_n, is given by*

$$\tau(T(H_n)) = \left[\frac{5n+1}{3}\right].$$

Proof By the definition of total graph, $V(T(H_n)) = V(H_n) \cup E(H_n)$ and $E(T(H_n)) = E(M(H_n)) \cup \{u_i v_i : 0 \leq i \leq n-1\} \cup \{v v_i : 0 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 0 \leq i \leq n-1$ subscripts modulo $n\}$. For each i, $(0 \leq i \leq n-1)$ the set of vertices $\{e_i, v_{i+1}, e_{i+1}, e'_{i+1}, e''_{i+1}\}$ of $T(H_n)$ induces a clique of order 5. Also the set of vertices $\{e'_i : 0 \leq i \leq n-1\} \cup \{v\}$ induces a clique K_{n+1} of order $n+1$. For each i, $(0 \leq i \leq n-1)$ the set of vertices $\{u_i, v_i, e''_i\}$ induces a 3-cycle in $T(H_n)$. Hence $\mathbb{C}_1 = \{u_i v_i e''_i : 0 \leq i \leq n-1\}$ is a set of vertex disjoint cycles of the subgraph of $T(H_n)$ induced by $\{u_i, v_i, e''_i : 0 \leq i \leq n-1\}$.



$M(H_9)$ and its vertex disjoint cycles

Figure 5.2

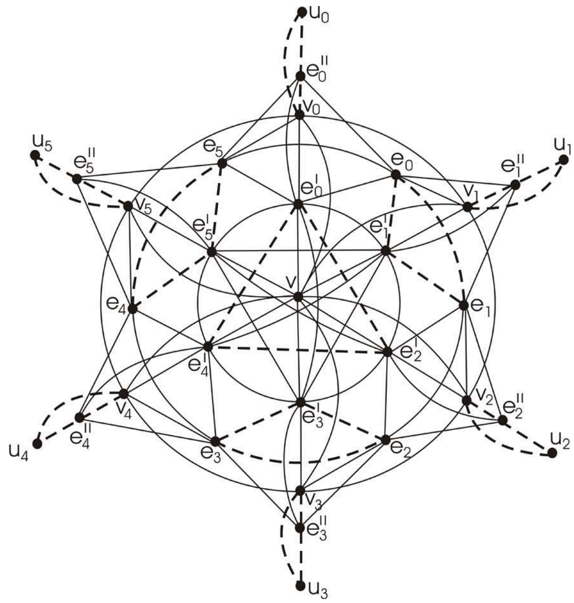
Case 1 *n is even.*

Let $\mathbb{C}_2$ be the collection of vertex disjoint 3-cycles of the subgraph induced by the set of vertices $\{e_i : 0 \le i \le n-1\} \cup \{e'_j : j = 2k+1, 0 \le k \le \dfrac{n}{2} - 1\}$. i.e., $\mathbb{C}_2 = \{e_i e_{i+1} e'_{i+1} : i = 2k, 0 \le k \le \frac{n}{2} - 1\}$. Let $\mathbb{C}_3$ be the set of 3-cycles of $T(H_n)$ induced by $\{e'_i : i = 2k, 0 \le k \le \frac{n}{2} - 1\} \cup \{v\}$. Since the subgraph induced by $\{e'_i : i = 2k, 0 \le k \le \frac{n}{2} - 1\} \cup \{v\}$ is a clique of order $\dfrac{n}{2} + 1$, $\mathbb{C}_3$ contains $\left[\dfrac{1}{3}\left(\dfrac{n}{2} + 1\right)\right]$ vertex disjoint 3-cycles. Since $V(\mathbb{C}_i) \cap V(\mathbb{C}_i) = \emptyset$ for $i \neq j$, $\tau(T(H_n)) \ge |\mathbb{C}_1| + |\mathbb{C}_2| + |\mathbb{C}_3| = \left[\dfrac{5n+1}{3}\right]$.

Case 2 *n is odd.*

Let $\mathbb{C}_2 = \{e_i e_{i+1} e'_{i+1} : i = 2k, 0 \le k \le \frac{n-3}{2}\}$ be the collection of vertex disjoint cycles of the subgraph induced by $\{e_i : 0 \le i \le n-2\} \cup \{e'_i : i = 2k+1, 0 \le k \le \frac{n-3}{2}\}$. Now $V' = V(T(H_n)) - \{V(\mathbb{C}_1) \cup V(\mathbb{C}_2)\} = \{e'_{2i} : 0 \le i \le \frac{n-1}{2}\} \cup \{e_{n-1}, v\}$ has $\dfrac{5n+1}{3}$ vertices and induced subgraph $\langle V' \rangle$ contains a clique of order $\dfrac{n+3}{2}$. If $\dfrac{n+3}{2} \equiv 0$ or $1(mod\ 3)$ then $\langle V' \rangle$ has $\left[\dfrac{1}{3}\left(\dfrac{n+5}{2}\right)\right]$ vertex disjoint 3-cycles disjoint from the cycles of $\mathbb{C}_1$ and $\mathbb{C}_2$.

If $\dfrac{n+3}{2} \equiv 2(mod\ 3)$ then $\left\langle \{e'_{2i} : 1 \le i \le \frac{n-3}{2}\} \cup \{v\} \right\rangle$ has $\dfrac{1}{3}\left(\dfrac{n-1}{2}\right)$ vertex disjoint 3-cycles and there exists another cycle $e_{n-1} e'_{n-1} e'_0$ disjoint from the cycles of $\mathbb{C}_1, \mathbb{C}_2$ and the cycles of $\left\langle \{e'_{2i} : 1 \le i \le \frac{n-1}{2}\} \cup \{v\} \right\rangle$. Hence in both the cases $\tau(T(H_n)) \ge |\mathbb{C}_1| + |\mathbb{C}_2| + \left[\dfrac{1}{3}\left(\dfrac{n+5}{2}\right)\right] = \left[\dfrac{5n+1}{3}\right]$. Since $|V(T(H_n))| = 5n+1$, it is clear that $\tau(T(H_n)) \le \left[\dfrac{5n+1}{3}\right]$. Hence $\tau(T(H_n)) = \left[\dfrac{5n+1}{3}\right]$. □



$T(H_6)$ and its vertex disjoint cycles

Figure 5.3

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